Squared-field amplitude modulus and radiation intensity nonequivalence within nonlinear slabs

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(Received 3 February 2005; published 25 May 2005)

This paper presents an approach to wave propagation inside the Fabry-Pérot framework. It states that the time-averaged Poynting vector modulus can be nonequivalent to the squared-field amplitude modulus. This fact permits the introduction of a kind of nonlinear medium whose nonlinearity is proportional to the time-averaged Poynting vector modulus. Its transmittance is calculated and found to differ from that obtained for a Kerr medium, whose nonlinearity is proportional to the squared-field amplitude modulus. The latter emphasizes the nonequivalence of these magnitudes. A space-time symmetry analysis shows that the Poynting nonlinearity should be possible only in noncentrosymmetric materials.

DOI: 10.1103/PhysRevE.71.056614 PACS number(s): 42.25.Bs, 42.65.Hw, 42.65.Pc

I. INTRODUCTION

A classical topic in electromagnetism is the study of wave transmission in a finite parallel-plane-face medium, known as a Fabry-Pérot resonator. When the medium presents nonlinear behavior, bistability appears [1]. To explain this phenomenon, the nonlinear Fabry-Pérot (NLFP) resonator was modeled by a third order susceptibility or Kerr-type nonlinearity [2]. At monochromatic plane wave excitation, the NLFP stationary regime is summarized in a non-time-dependent nonlinear wave equation for the complex field amplitudes, the nonlinear Helmholtz equation (NLHE). Hence, the reflectance and transmittance problem reduces to finding the NLHE solution with appropriate boundary conditions for the field amplitude modulus and phase.

The NLHE complexity led to approximate methods of resolution. Many approaches consider two counterpropagating waves in the medium and the analysis is done by separately considering the effects on each wave [2–5]. Unfortunately, the linear superposition principle is no longer valid in nonlinear media and the separation of the electromagnetic field in these back and forth waves is meaningless. As a result, the NLHE separation into two equations, one for each wave, is only possible by neglecting various coupling nonlinear terms that would give an important contribution to the accuracy of the transmittance results. Moreover the slowly varying envelope approximation (SVEA) is often applied to these waves [2–4] but, within the counterpropagating wave approach, its validity was questioned [5]. Also, the boundary conditions were simplified rather than rigorously treated [2,6]. The above facts suggest that all these approximated approaches cannot physically be equivalent to the exact problem.

The work done by Chen and Mills exactly solved the NLFP problem for an absorptionless Kerr-type medium [7]. The proper resolution method was to assume a general complex field within the medium, disregarding the concept of counterpropagating waves. Chen and Mills derive a two-coupled-equation system for the field amplitude modulus and phase together with general boundary conditions, thus obtaining an analytic-transcendental solution for the transmittance of the NLFP resonator.

On the other hand, their work permitted us to note an implicit difference between the time-averaged Poynting vector modulus, i.e., the electromagnetic radiation intensity I and the squared-field amplitude modulus ($|E|^2$) inside the nonlinear medium. If the nonequivalence of these magnitudes were true, it could change certain well-established fundamental concepts in classical electrodynamics. This fact motivated us to develop a different approach to wave propagation in nonlinear media inside the Fabry-Pérot framework, called the *S formalism*. It introduces a variable related to the time-averaged Poynting vector which states that its magnitude can be nonequivalent to the squared-field amplitude modulus, contrary to the common usage. Furthermore, the S formalism presents two important advantages: it permits one to directly monitor the radiation intensity within the medium, and it avoids approximations, such as the SVEA, simplification of boundary conditions, and so on.

The fact that the time-averaged Poynting vector modulus is nonequivalent to the squared-field amplitude modulus, as the S formalism will show, implies that the nonlinearity of Kerr-type media is not proportional to the intensity, which is contrary to what has been established to date. This assertion leads to the following question regarding the modelling of the NLFP resonator: Is it a Kerr-type nonlinearity, or does it vary proportionately to the intensity? As this question does not have a definitive answer, the existence of the latter cannot be denied. Then, we define the *Poynting medium* as a medium where the nonlinearity is proportional to the intensity. Thus, our objective is to solve the Poynting NLFP problem through the S formalism, comparing the resultant transmittance with that obtained for a Kerr NLFP resonator to remark the nonequivalence between squared-field amplitude modulus and radiation intensity.

In Sec. II, we derive the S formalism in the following form. First the time-averaged Poynting vector assuming har-

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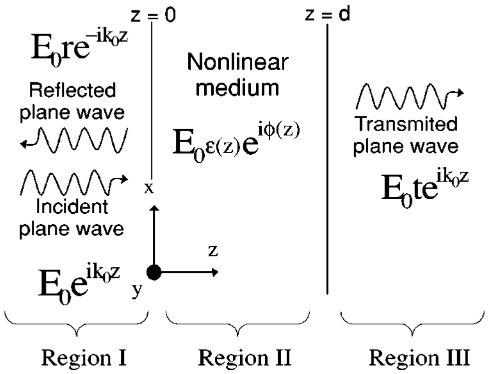

FIG. 1. A harmonic plane wave strikes a nonlinear Fabry-Pérot resonator, to be reflected and transmitted. Regions I and III constitute, for simplicity, the same linear dielectric medium (e.g., air).

monic fields is calculated. Then, a dimensionless variable that is proportional to intensity is introduced. Thus, a set of field evolution differential equations and the general boundary condition equations on this field variable are obtained, which constitute the S formalism. In Sec. III, our approach is applied to derive the transmittance for the Poynting NLFP resonator and the results compared with those obtained for a Kerr NLFP resonator. A brief discussion about the possibility of existence and observation of Poynting media is given. Finally, in Sec. IV, we conclude.

II. THE *S*-FORMALISM APPROACH

Referring to Fig. 1, we start writing the linearly polarized transversal harmonic electromagnetic fields of frequency ω as

$$\mathbf{E}_\ell(z,t) = \tfrac{1}{2}[E_\ell^\omega(z)e^{-i\omega t} + \text{c.c.}]\hat{\mathbf{i}}, \tag{1a}$$

$$\mathbf{H}_\ell(z,t) = \tfrac{1}{2}[H_\ell^\omega(z)e^{-i\omega t} + \text{c.c.}]\hat{\mathbf{j}}, \tag{1b}$$

where $E_\ell^\omega(z)$ and $H_\ell^\omega(z)$ are the non-time-dependent complex amplitudes for $\ell = \text{I},\text{II},\text{III}$. From region I, a plane wave of amplitude E_0 and wave vector k_0 impinges perpendicularly on a nonmagnetic, isotropic, and spatially nondispersive medium of thickness d (region II). The optical field is assumed to maintain its polarization along this region so that a scalar approach is valid. The reflected and transmitted plane waves have amplitudes rE_0 and tE_0 with r and t the complex reflection and transmission coefficients, respectively. Then, in regions I and III the spatially-dependent complex amplitudes are given by

$$E_\text{I}^\omega(z) = E_0(e^{ik_0 z} + re^{-ik_0 z}), \tag{2}$$

$$E_\text{III}^\omega(z) = E_0 te^{ik_0 z}. \tag{3}$$

Similarly to Ref. [7], in region II, we write down the following ansatz for the spatially-dependent complex amplitude of the electric field:

$$E_\text{II}^\omega(z) = E_0 \mathcal{E}(z) e^{i\phi(z)}, \tag{4}$$

where the dimensionless amplitude $\mathcal{E}(z)$ and phase $\phi(z)$ are both real functions of z.

The time-averaged Poynting vector $\langle \mathbf{E}_\ell(z,t) \times \mathbf{H}_\ell(z,t) \rangle$ can be easily calculated with the aid of Faraday's law, giving

$$\langle \mathbf{S} \rangle_\ell = \frac{1}{2\mu_0 \omega} \,\text{Im}\left\{ [E_\ell^\omega(z)]^* \frac{\partial E_\ell^\omega(z)}{\partial z} \right\} \hat{\mathbf{k}}, \tag{5}$$

where μ_0 is the vacuum permeability. From this expression we calculate the intensities for the three regions:

$$\langle S \rangle_\text{I} = I_0(1 - |r|^2), \tag{6a}$$

$$\langle S \rangle_\text{II} = I_0 k_0^{-1} \mathcal{E}^2(z) \frac{\partial \phi(z)}{\partial z} \equiv I_0 S(z), \tag{6b}$$

$$\langle S \rangle_\text{III} = I_0 |t|^2, \tag{6c}$$

where $I_0 = k_0 E_0^2 / (2\mu_0 \omega)$ is the incident intensity. In region II, Eq. (6b) defines the dimensionless field variable

$$S \equiv k_0^{-1} \mathcal{E}^2 \frac{\partial \phi}{\partial z}, \tag{7}$$

directly related to the intensity inside the medium, which will characterize the S formalism. From Eq. (7) it is clear that if ϕ is not a linear function of z, as often happens in nonlinear media, then S and $\mathcal{E}^2$ are nonequivalent.

The next step is to derive the NLHE in terms of the classical field variables $(\mathcal{E},\phi)$, and transform it into a set of equivalent equations in terms of $(\mathcal{E},S)$. The NLHE is derived from the macroscopic Maxwell equations complemented by appropriate constitutive relations. We assume that the polarization $\mathbf{P}$ and current density $\mathbf{J}$ vary only in the electric field direction with frequency ω, neglecting higher harmonics, and their spatially-dependent complex amplitudes satisfy the following constitutive relations:

$$P_\text{II}^\omega(z) = \epsilon_0 \chi_\text{gen}[z, E_\text{II}^\omega, H_\text{II}^\omega] E_\text{II}^\omega(z), \tag{8}$$

$$J_\text{II}^\omega(z) = \sigma_\text{gen}[z, E_\text{II}^\omega, H_\text{II}^\omega] E_\text{II}^\omega(z), \tag{9}$$

where ϵ_0 is the vacuum permittivity and χ_gen and σ_gen are the generalized susceptibility and conductivity, respectively, which are real and contain the linear as well as a possible nonlinear medium response. Note that the constitutive relations are not explicitly written since cases could exist where it is not possible to describe the nonlinear polarization and current density by the classical electric field power expansion. Thereby, the scalar NLHE is

$$\left[\frac{d^2}{dz^2} + k_0^2(1 + \chi_\text{gen}) + i\omega\mu_0\sigma_\text{gen} \right] E_\text{II}^\omega(z) = 0. \tag{10}$$

This equation constitutes the starting point to study several linear and nonlinear monochromatic wave propagation phenomena within the Fabry-Pérot framework. Substituting Eq. (4) into Eq. (10) and using Eq. (7), we derive the following set of spatial evolution equations for the field variables $\mathcal{E}(z)$ and $S(z)$:

$$\frac{d^2 \mathcal{E}}{dz^2} + k_0^2 \left((1 + \chi_\text{gen}[z, \mathcal{E}, S]) \mathcal{E} - \frac{S^2}{\mathcal{E}^3} \right) = 0, \tag{11a}$$

$$\frac{dS}{dz} + \frac{\omega}{k_0}\mu_0\sigma_{\rm gen}[z,\mathcal{E},S]\mathcal{E}^2 = 0. \quad (11b)$$

To guarantee the physical content of the solution, these equations must be necessarily complemented with the following boundary conditions: the continuity of the tangential components of the electric and magnetic field at the interfaces. The general boundary conditions were rigorously derived in Ref. [7]: four equations as functions of $(\mathcal{E},\phi)$ at $z=0$ and d which, by using Eq. (7), are transformed into three equations in terms of the variables $(\mathcal{E},S)$ to give

$$\left(\mathcal{E}(0) + \frac{S(0)}{\mathcal{E}(0)}\right)^2 + \left(\frac{1}{k_0}\frac{d\mathcal{E}}{dz}\bigg|_{z=0}\right)^2 = 4, \quad (12a)$$

$$S(d) - \mathcal{E}^2(d) = 0, \quad (12b)$$

$$\frac{d\mathcal{E}}{dz}\bigg|_{z=d} = 0. \quad (12c)$$

From Eqs. (6) and (12), the transmittance is obtained as

$$T = |t|^2 = S(d), \quad (13)$$

and the energy conservation is guaranteed thorough the expression

$$|r|^2 + |t|^2 = 1 - [S(0) - S(d)]. \quad (14)$$

This equation establishes that the reflectance and transmittance are limited by the boundary values of the time-averaged Poynting vector. For a nonabsorbent medium $S(d)=S(0)$, then $|r|^2+|t|^2=1$.

Equations (11) and (12) represent the S formalism; they were derived without assuming approximations such as counterpropagating waves, the SVEA, simplifications of the boundary conditions, and so on. Also, note that Eq. (11b) represents the time-averaged Poynting theorem applied to the problem of harmonic fields simplifying the interpretation of $\sigma_{\rm gen}$ as the dissipation properties of the medium. In particular, when $\sigma_{\rm gen}=0$, the dimensionless intensity S is a constant fixed by the boundary conditions. Furthermore, through $S(z)$ it is possible to monitor directly the intensity along the medium as a function of the spatial coordinate, as opposed to using the conventional formalism.

The S formalism is useful to analyze the linear case as well as the nonlinear one. Before studying the latter, i.e., the comparison between the Poynting and Kerr media in an effort to show the explicit difference between S (or I) and $\mathcal{E}^2$ (or $|E^2|$) in nonlinear media, we refer to the linear case. According to our analysis [8], there are only two situations where the relationship $I=cte|E^2|$ holds true. First, a single plane wave propagates in an infinite or semi-infinite linear dielectric characterized by $\sigma_{\rm gen}=0$ and $\chi_{\rm gen}=\chi^{(1)}$ where $\chi^{(1)}$ is the linear susceptibility. Under these conditions, Eqs. (11) relate the constants S and $\mathcal{E}$ by $S=(1+\chi^{(1)})^{1/2}\ \mathcal{E}^2$. Second, a single plane wave propagates in a semi-infinite linear absorber characterized by $\chi_{\rm gen}=\chi^{(1)}$ and $\sigma_{\rm gen}=\sigma$ where σ is the Ohmic conductivity and such that $S(z)\propto\mathcal{E}^2(z)$, both being proportional to a decreasing exponential function of z. On the contrary, when the medium is finite, e.g., a Fabry-Pérot resonator with boundary conditions at the interfaces, S is no longer equivalent to $\mathcal{E}^2$, not even for the linear dielectric case, because S is a constant and $\mathcal{E}^2$ is an oscillating function of z [8].

III. THE POYNTING MEDIUM

A. Constitutive relations and transmittance results

At this point, we introduce the Poynting medium by the following constitutive relations:

$$\chi_{\rm gen} = \chi^{(1)} + \gamma I_0 S(z), \quad (15)$$

$$\sigma_{\rm gen} = 0, \quad (16)$$

where γ is the nonlinear coefficient. Equations (11) have a simple analytical solution given by

$$S(z) = S_0, \quad (17)$$

$$\mathcal{E}(z) = \sqrt{\frac{S_0}{2}\left[\left(1 - \frac{k_0^2}{k_1^2}\right)\cos[2k_1(z-d)] + 1 + \frac{k_0^2}{k_1^2}\right]}, \quad (18)$$

where $\gamma > 0$ and $k_1^2=k_0^2(1+\chi^{(1)}+\gamma I_0 S_0)$. The constant S_0 is fixed by

$$\left(1 - \frac{k_1^2}{k_0^2}\right)\mathcal{E}^2(0) + \left(3 + \frac{k_1^2}{k_0^2}\right)S_0 - 4 = 0. \quad (19)$$

Combining Eqs. (18) and (19), the transmitance can be expressed in a similar fashion as for the linear Fabry-Pérot resonator as

$$T = \frac{1}{1 + F\sin^2(k_1 d)}, \quad (20)$$

where $F=k_0(1-k_1/k_0)^2/(4k_1)$, carefully noting that Eq. (20) is a transcendental expression since k_1 depends on S_0.

Now, we compare the transmittance results for the Poynting and Kerr media. The latter are defined by

$$\chi_{\rm gen} = \chi^{(1)} + \gamma I_0 \mathcal{E}^2(z), \quad (21)$$

$$\sigma_{\rm gen} = 0. \quad (22)$$

The Kerr NLFP transmittance results were taken from Ref. [7]. Figure 2 shows T against the nonlinear parameter γI_0 for two different values of $\chi^{(1)}$. Figures 2(a_i) and 2(b_i) correspond to the Poynting and Kerr medium, respectively. From these figures, it is apparent that the transmittance of the Poynting NLFP as well as of the Kerr NLFP resonator is multistable. However, for increasing values of $\chi^{(1)}$, the peak transmittance separation diminishes for the Kerr medium while it increases for the Poynting medium. Also, the Kerr multistability appears for the smallest values of the nonlinear parameter γI_0. The transmittance difference of the two media emphasizes the I and $|E|^2$ nonequivalence. Figure 3 depicts the dependence of T on the dimensionless thickness $k_0 d(1+\chi^{(1)})^{1/2}/(2\pi)$ enhancing the nonlinearity difference of the Polynting and Kerr media. Note that the departure from an Airy-type function for the Kerr medium is stronger than that for the Poynting medium.

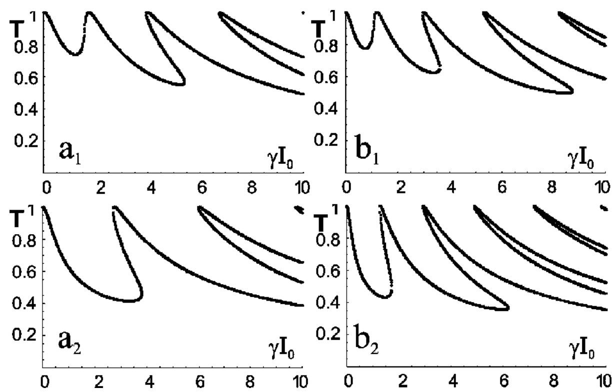

FIG. 2. Transmittance against nonlinear parameter for (a_i) Poynting and (b_i) Kerr medium with $k_0d=2\pi$. For $i=1$, $\chi^{(1)}=1.25$; and $i=2$, $\chi^{(1)}=5.25$.

On the other hand, Fig. 4 shows that the Poynting nonlinear susceptibility $\chi_{gen}-\chi^{(1)}$ has a constant value along the medium (z coordinate). In return, the Kerr nonlinear susceptibility varies periodically. This fact implies the formation of a phase grating in the Kerr medium in contrast to the Poynting medium. Perhaps this substantial difference can be measured, and this could be the starting point to experimentally identify a Poynting medium.

B. Transformation properties under spatial inversion and time reversal

It is a fact that several unusual types of nonlinearities were predicted before its experimental observation, as was remarked, for example, in the pioneer works of Baranova *et al.* [9]. With the aim to elucidate the isotropic medium requirements to observe these phenomena, those authors pointed out the necessity of an analysis of transformation properties of electromagnetic quantities under rotation, spatial inversion, and time reversal. Therefore, this symmetry analysis is also necessary to delimit the Poynting medium requirements.

The magnitude that characterizes the electromagnetic response of a Poynting medium is their nonlinear susceptibility $\chi^{(P)}$ which is linear on the time-averaged Poynting vector $\langle \mathbf{S} \rangle$, as follows from the constitutive relations [Eq. (15)]. In a general form, it can be written as

$$\chi_{ij}^{(P)} = \gamma_{ijk}\langle \mathbf{S}(\mathbf{r},t)\rangle_k, \qquad (23)$$

with $i,j,k=x,y,z$ and

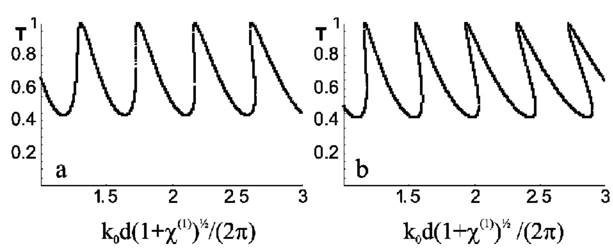

FIG. 3. Transmittance against dimensionless thickness for (a) Poynting and (b) Kerr medium with $\chi^{(1)}=5.25$ and $\gamma I_0=2$.

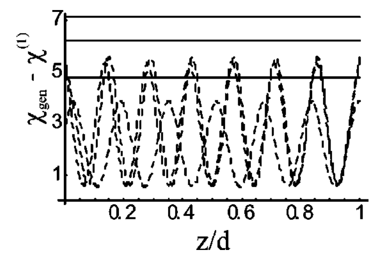

FIG. 4. Nonlinear susceptibility against dimensionless spatial coordinate for each of the three solutions compatible with the boundary conditions. Continuous line: Poynting medium. Broken line: Kerr medium. The parameter values are $\gamma I_0=9$, $\chi^{(1)}=1.25$, and k_0d as defined in Fig. 2.

$$\langle \mathbf{S}(\mathbf{r},t)\rangle_k = \frac{1}{T}\int_t^{t+T} [\mathbf{E}(\mathbf{r},t') \times \mathbf{H}(\mathbf{r},t')]_k dt', \qquad (24)$$

where $\mathbf{E}$ and $\mathbf{H}$ are harmonics of period $\tau=2\pi/\omega$ and time interval $T \gg \tau$. The susceptibility tensor transforms as even under spacial inversion $\mathbf{r}\rightarrow-\mathbf{r}$ and time reversal $t\rightarrow-t$, contrary to the Poynting vector and its time-averaged value, which transform as odd under spatial inversion and time reversal, i.e., $\langle \mathbf{S}(\mathbf{r},t)\rangle \rightarrow -\langle \mathbf{S}(-\mathbf{r},t)\rangle$ and $\langle \mathbf{S}(\mathbf{r},t)\rangle \rightarrow -\langle \mathbf{S}(\mathbf{r},-t)\rangle$, respectively [10]. Then, a medium possessing a linear connection between χ_{ij} and $\langle S\rangle_z$ should be *noninvariant* with respect to spatial inversion and time reversal. Otherwise, the space-time symmetry will be violated in the constitutive relation [Eq. (23)].

The lack of parity symmetry under inversion of coordinates is a property of materials without inversion center, i.e., Poynting nonlinearity should be only possible in *noncentrosymmetric* materials. There are several material candidates to possess a Poynting nonlinearity, such as, for example, cubic crystals with zinc-blende structure like GaAs, InSb, and others. In these materials intensity-dependent transmission and bistability were experimentally observed [11]. Also, isotropic homogeneous liquids formed by nonracemic mixtures or solutions of mirror-asymmetric (chiral) molecules with strong nonlinear optical susceptibility, as products of several nonlinear processes [12], are also feasible systems to possess a Poynting nonlinearity. In addition, parity under time reversal should be violated in Poynting media. This means that weak dissipative process that converts field energy into heat is necessary to remove the rule relating to the $t\rightarrow-t$ transformation. For example, either very weak absorbtion or current flow caused by an external quasistatic field, which basically do not affect the wave propagation at light frequency ω, would ensure medium noninvariance under time reversal.

We believe that, although experimental work is required, the above preliminary analysis could stimulate further discussion regarding the existence of Poynting media.

IV. CONCLUSIONS

In summary, we derived a formalism in terms of dimensionless variables related to the time-averaged Poynting vector and field amplitude modulus within the Fabry-Pérot framework. The S formalism shows explicitly that the energy intensity and squared-field amplitude modulus are only equivalents for a single plane wave propagating in a linear infinite or semi-infinite medium. Otherwise, they are non-equivalent. Additionally, the *S* formalism presents two important advantages: it permits one to directly monitor the time-averaged Poynting vector in the medium and it avoids approximations, such as the SVEA, simplification of the boundary conditions, and so on. To emphasize this non-equivalence we introduce the Poynting medium, whose nonlinearity is proportional to the intensity instead of the electric squared-field amplitude modulus as in the Kerr medium. We find marked disagreement in the transmittance of the two media, which support the differences between I and $|E|^2$. Also, a space-time symmetry analysis shows that the Poynting nonlinearity should only be possible in noncentrosymmetric materials.

The statements and analysis pointed out here constitute an advance on theoretical views of basic concepts in electrodynamics. The *S* formalism could be important in problems where the time-averaged Poynting vector must be rigorously monitored as in photoconductor or photorefractive materials. Further to this particular case studied here, this approach leaves open the possibility of physical results in actual topics on nonlinear wave propagation such as spatial solitons, wave mixing, and others. Finally, we leave open the possibility that experimental techniques, based on intensity-dependent phase changes of a Gaussian beam such as the *Z*-scan technique [13], might not truly measure Kerr-type nonlinearity. On the contrary, they could be measuring a Poynting-type nonlinearity instead.

ACKNOWLEDGMENTS

The authors thank Professor Boris Ya. Zel'dovich for suggestions about symmetry properties of Poynting media. The authors also thank Victor Waveluk for valuable advice. A.L. acknowledges CLAF-CNPq for financial support.

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